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Transverse Effects and Noise in Optical Instabilities

L. A. Lugiato, R. J. Horowicz, G. Strini and L. M. Narducci

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Transverse effects and noise in optical instabilities

BY L. A. LUGIATO[1], R. J. HOROWICZ[1]†, G. STRINI[1] AND L. M. NARDUCCI[2]
[1] *Dipartimento di Fisica dell'Università, Via Celoria* 16, 20133 *Milano, Italy*
[2] *Department of Physics, Drexel University, Philadelphia, Pennsylvania* 19104, *U.S.A.*

We analyse the simplest model that describes the dynamics of a two-level, optically bistable system in a ring cavity, and incorporates a Gaussian radial variation of the electric field. We find that the most favourable situation to achieve self-pulsing instabilities is that of atomic and cavity detunings with opposite sign. The results are compared with those of plane-wave theory. We show that noise plays a primary role at the onset of self-oscillatory instabilities, and that its effects are governed by an equation formally identical to the Risken equation for the laser.

1. INTRODUCTION

The detailed comparison between theory and experiment in the study of quantum optical systems imposes a thorough analysis of the effects that arise from the radial variation of the electric field in the radiation beam (transverse effects). This is especially true in the case of optical bistability (o.b.), in which the modelling of practical devices requires precise predictions on the behaviour of the system.

From the viewpoint of theory, this implies a renunciation of the comfortable ground of plane-wave theory and coping with the difficulties of two-dimensional theories. From the experimental viewpoint it is necessary to perform detailed and systematic observations of the behaviour of the radial profile, an aspect that is most often disregarded. This is necessary to allow a comparison with theoretical predictions, and to help theory in attaining an adequate description of the physical situation.

In recent years, considerable efforts have been devoted to the study of transverse effects (see, for example, Bowden *et al.* (eds) (1984). A few of these works analyse the transverse effects on instabilities and spontaneous pulsations in o.b. (Moloney *et al.* 1982; Firth & Wright 1982; Lugiato & Milani 1983) or in the laser (Hauck *et al.* 1983). These results suggest that the instability problem including transverse effects is not simply an extension or a generalization of the plane-wave case, but rather an independent piece of physics. In fact, there are examples of plane-wave instabilities that disappear in the Gaussian case (Lugiato & Milani 1983), as well as of instabilities predicted by a full two-dimensional treatment, that are absent in plane-wave theory (Hauck *et al.* 1983).

In this paper we study the transverse effects on a class of instabilities in mixed absorptive and dispersive o.b., previously analysed in the framework of plane-wave theory by Lugiato *et al.* (1982) and Lugiato & Narducci (1984); see also Ikeda & Akimoto (1982). Our treatment is based on the simplest possible model that describes the dynamics of this system and incorporates a radial variation of the electric field. This model, which allows for an analytical

† Permanent address: Departamento de Fisica, Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro, Brazil.

study of the stability of the steady-state solutions, is described in §2, in which the stationary solutions are also calculated. The results of the linear stability analysis are illustrated, and compared with those of plane-wave theory, in §3.

Another general class of effects that must be carefully studied for practical purposes are those that arise from noise. Fluctuation phenomena in o.b. have been extensively analysed from a theoretical viewpoint, but almost exclusively in the stationary state (Lugiato 1984). We recently studied some noise effects in the transient or in the stable self-pulsing behaviour, in the critical situations in which these effects are most visible. Another paper in this symposium, (by Broggi & Lugiato), illustrates the role of fluctuations in the critical slowing down, and shows that they give rise to a broad switching-time distribution, as well as an interesting effect of transient noise-induced optical bistability. On the other hand, in §4 of this paper we discuss the effects of noise in the onset of self-pulsing instabilities. The concluding discussion is given in §5.

2. Gaussian one-mode model for optical bistability

We consider a unidirectional ring cavity with spherical mirrors, of total length $\mathscr{L}$. It contains a cylindrical, homogeneously broadened two-level atomic sample of length L and radius d. We assume that the Fresnel number, $w_0^2/\lambda L$, where w_0 is the beam waist and λ the wavelength, is so large that the beam radius is practically constant along the atomic sample. The incident field is assumed to be matched to the fundamental TEM_{00} mode of the cavity.

If we indicate by $\bar{r}$ the radial variable normalized to the beam waist, the one-mode model that we study is given by the dynamical equations (Lugiato & Milani 1983)

$$k^{-1}\frac{\mathrm{d}}{\mathrm{d}t}f(t) = -f(1+\mathrm{i}\theta)+y-2C\int_0^{d/w_0}\mathrm{d}\bar{r}\,4\bar{r}\exp(-\bar{r}^2)\,P(\bar{r},t), \tag{1a}$$

$$\gamma_\perp^{-1}\frac{\partial}{\partial t}P(\bar{r},t) = D(\bar{r},t)f(t)\exp(-\bar{r}^2)-(1+\mathrm{i}\Delta)\,P(\bar{r},t), \tag{1b}$$

$$\gamma_\parallel^{-1}\frac{\partial}{\partial t}D(\bar{r},t) = -\tfrac{1}{2}\{P(\bar{r},t)f^*(t)+P^*(\bar{r},t)f(t)\}\exp(-\bar{r}^2)-D(\bar{r},t)+1, \tag{1c}$$

where f and y are the normalized amplitudes of the output and incident field, respectively; $P(\bar{r},t)$ and $D(\bar{r},t)$ are normalized quantities that correspond to the macroscopic atomic polarization and population difference at a distance r from the longitudinal axis. The field damping constant, k, is defined as $cT/\mathscr{L}$, where T is the mirror transmissivity coefficient. The atomic decay rates, $\gamma_\parallel$ and $\gamma_\perp$, are the inverse of the relaxation times T_1 and T_2', respectively. The bistability parameter, C, is usually defined as

$$C = \alpha L/2T, \tag{2}$$

where α is the unsaturated field absorption coefficient. The cavity and atomic detunings θ and Δ are given by

$$\theta = (\omega_c-\omega_0)/k, \quad \Delta = (\omega_a-\omega_0)/\gamma_\perp, \tag{3}$$

where ω_0, ω_c and ω_a are respectively the frequency of the incident field, the frequency of the cavity mode and the atomic transition frequency. The model (1) incorporates the following assumptions: (i) the mean field limit $\alpha L \ll 1$, $T \ll 1$, with $C = \alpha L/2T$ arbitrary; (ii) the time evolution occurs on a timescale much longer than the cavity transit time; and (iii) the transverse profile of the electric field inside the filled cavity corresponds to the Gaussian TEM_{00} mode.

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This one-transverse-mode assumption holds in at least two situations: (1) when all the other modes are detuned enough from the input frequency, ω_0, and (2) when the losses of all the other transverse modes are large enough to maintain their amplitude negligible.

At steady state, one obtains from (1) the equation that links the input and output fields:

$$y^2 = x^2[\{1+2Cg(x^2)\}+\{\theta-2C\Delta g(x^2)\}], \tag{4}$$

where we define

$$x = |f_{st}|$$

and

$$g(x^2) = \frac{1}{x^2}\ln\frac{1+\Delta^2+x^2}{1+\Delta^2+x^2\exp\{-2(d/w_0)^2\}}. \tag{5}$$

This state equation was derived by Ballagh *et al.* (1981), Arimondo *et al.* (1981), Drummond (1981) and Lugiato & Milani (1983). In the limit $d/w_0 \to 0$ one recovers the plane-wave theory (Lugiato & Milani 1983). In the following, we shall always consider the opposite case $d/w_0 \to \infty$.

3. Linear stability analysis

The details of the stability analysis of the steady-state solutions can be found in Lugiato *et al.* 1984*b*). Here we limit ourselves to the discussion of the results.

First, let us recall the picture in the plane-wave theory (Lugiato *et al.* 1982). For suitable choices of the parameters C, Δ, θ, $\tilde{k} \equiv k/\gamma_\perp$ and $\tilde{\gamma} = \gamma_\parallel/\gamma_\perp$, one finds that a portion of the steady-state curve with positive slope is unstable. A necessary condition is that Δ or θ or both are non-zero. With the possible exception of a small range of values of the incident field in correspondence with which the long-term behaviour exhibits precipitation to the lower transmission state, this instability leads to undamped self-pulsing behaviour, with an oscillation period on the order of the cavity build-up time, k^{-1}. When C is very large, the instability range includes a domain of chaotic behaviour. On approaching this domain on either side, one identifies a cascade of period-doubling bifurcations of the usual type.

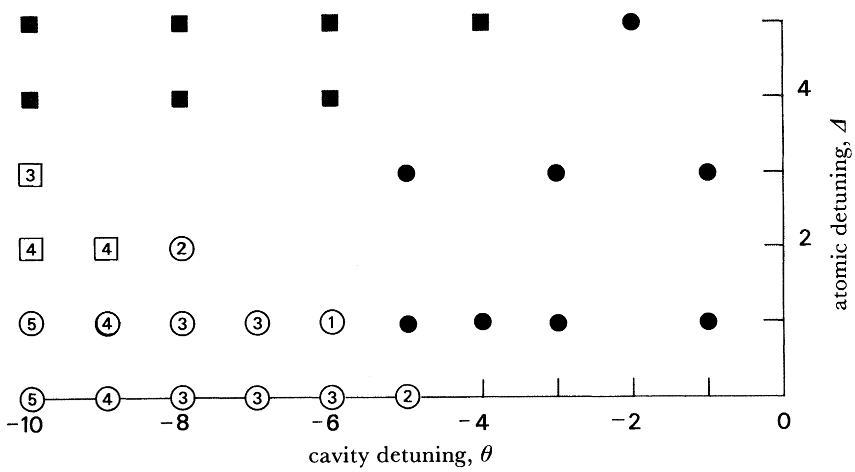


Figure 1. Positive-slope instability domain in the Δ–θ plane, in the plane-wave case. $C = 75$, $\tilde{k} = 0.5$, $\tilde{\gamma} = 2$. Filled circles indicate that the steady-state curve is S-shaped and that no positive-slope instability is there. Filled squares indicate that there is neither bistability nor instability. Open circles indicate that the steady-state curve is S-shaped and that there is a positive slope instability range, the approximate extension of which in the variable x is shown in the circle. Open squares indicate that there is no bistability, but the steady-state curve includes an instability range, the approximate extent of which is indicated in the square.

We analysed the stability of the steady-state curve for $C = 75$, $\tilde{k} = 0.5$, $\tilde{\gamma} = 2$, $-10 \leqslant \Delta$, $\theta \leqslant 10$. Figure 1 shows the extension of the positive-slope instability region in the second quadrant of the Δ–θ plane in the plane-wave case. No instability was found in the first quadrant, provided that Δ was not too small. A simultaneous change of sign in Δ and θ does not change anything with respect to the stability of the stationary states. If we increase the value of $\tilde{k}$ by keeping fixed the other parameters C, Δ, θ and $\tilde{\gamma}$, the unstable range in the steady-state curve of transmitted against incident field grows.

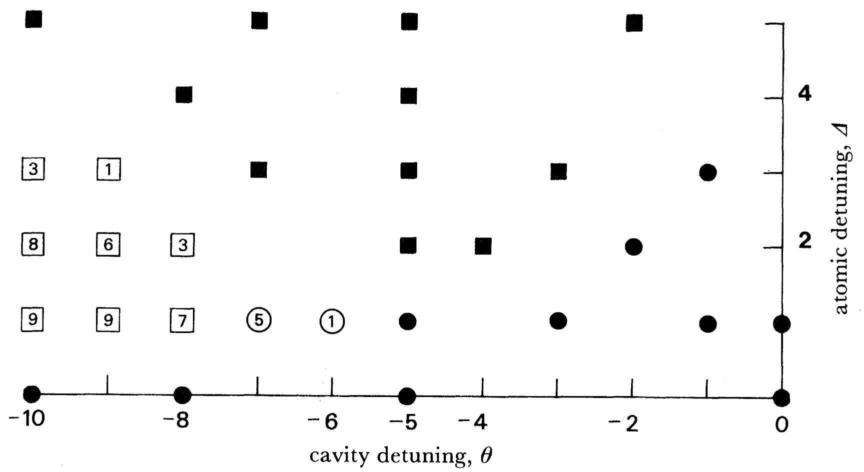
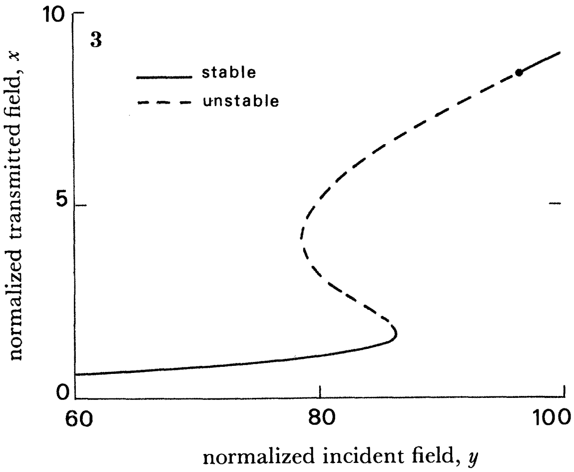


FIGURE 2. As for figure 1, but in the Gaussian case with $d/w_0 = \infty$.

Let us now turn to the Gaussian case. Figure 2 exhibits the extension of the positive slope instability region for the same values of the parameters considered in figure 1. Again, we do not find any positive slope instability in the first quadrant of the Δ–θ plane and in the domain $5 \leqslant \Delta \leqslant 10$, $-10 \leqslant \theta \leqslant 10$. Hence for $C \lesssim 100$ the situation $\Delta\theta < 0$, which is less favourable for bistability than the case $\Delta\theta > 0$, is on the contrary necessary to obtain a self-pulsing instability. The position of the instability domain in the Δ–θ plane is substantially the same as in the plane-wave case; however, we do not find any positive slope instability in the purely absorptive situation $\Delta = 0$. Figure 2 indicates also that in the Gaussian case the situation of S-shaped steady-state curve is less favourable to instabilities than that of a single-valued curve. On comparing the data in figures 1 and 2 for the same values of Δ and θ, we see that, for Δ not too small, the positive-slope instability range in the variable x turns out to be larger in the Gaussian case. As we see from figures 3 and 4, this feature is mainly due to the fact that in the plane-wave case a sizable part of the instability range has negative slope. On the other hand, the situation of the single-valued steady-state curve is ideal for pulsation, because we are sure *a priori* that in the whole instability range the system exhibits undamped self-pulsing. In fact, in this case no competitive process of long-term precipitation to the lower transmission branch is possible.

An interesting quantity is the ratio (instability range)/(instability threshold), where both the instability range and the instability threshold refer to the input field variable y. When the steady-state curve is S-shaped, the instability threshold is the minimum value of y for which the higher transmission steady state is unstable. In figures 3 and 4 the values of this ratio are practically identical and equal to 0.23.



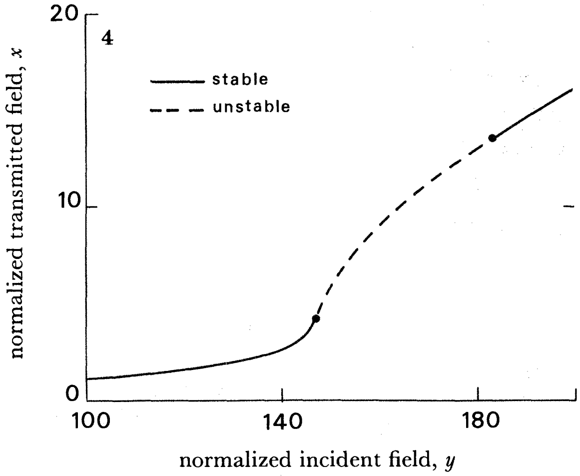


FIGURE 3. Plane-wave case, $C = 75, \Delta = 1, \theta = -9, \tilde{k} = 0.5, \tilde{\gamma} = 2$. The steady-state curve of normalized transmitted field, x, is shown as a function of normalized incident field, y, and the part of the curve that is unstable against fluctuations is indicated.

FIGURE 4. As for figure 3, but in the Gaussian case with $d/w_0 = \infty$.

We are currently analysing the numerical solutions of the full nonlinear time-dependent dynamical equations (1), to see what kind of oscillations arise from this instability in the Gaussian case.

4. Noise effects in the onset of self-pulsing instabilities

The instability discussed in the previous section arises in the cavity mode nearest to resonance with the incident field (resonant mode), which is the only mode that is taken into account in (1). On the other hand, the first instability that was discovered in the Maxwell–Bloch equations for o.b. arises in the off-resonance longitudinal cavity modes, adjacent to the resonant mode (Bonifacio & Lugiato 1978; Lugiato 1980). Hence in this case the instability must be described by a many-mode theory and leads to spontaneous oscillations with a period on the order of the cavity round-trip time. The slowly varying envelope of the electric field does not oscillate uniformly in the cavity, as in the pulsations arising from the instability described in the previous section, but exhibits a wave profile that propagates and circulates along the cavity.

In Lugiato & Milani (1983) it is shown that in the purely absorptive case this off-resonance mode instability disappears when the electric field has a Gaussian transverse profile. On the other hand, a recent paper (Carmichael *et al.* 1984) demonstrates that in the general mixed absorptive and dispersive case the instability remains, and sometimes is even enhanced with respect to the plane-wave situation.

We analyse the effects of thermal, external and quantum noise on the onset of the off-resonance mode instability in o.b. (Lugiato *et al.* 1984*a*). For the sake of simplicity, we consider the plane-wave, purely absorptive case. Use of the dressed-mode formalism of o.b. (Benza & Lugiato 1981) turns out to be essential. In Benza & Lugiato (1981) the analysis of noise effects was restricted to the region below threshold, where a simple linearized treatment is possible. Thus, the critical slowing down and the spectral line narrowing effects in the approach to the instability threshold were described. Here we extend this analysis by including the nonlinear terms that become essential in the threshold region. This makes it possible to achieve a full description of fluctuations at the onset of instability.

It was repeatedly claimed (see, for example, Gronchi *et al.* 1981) that the behaviour of the adjacent cavity modes at the instability threshold is completely analogous to that of the laser field at laser threshold. In fact, these modes become unstable when the side-mode gain becomes larger than the loss. Furthermore, at instability threshold the adjacent modes spring up from zero under the triggering of fluctuations, with a random phase. Our present analysis substantiates this picture decisively. In fact, let us call β_1 and β_{-1} the complex amplitudes of the two side-modes. Let $P(\beta_1, \beta_1^*, t)$ and $P(\beta_{-1}, \beta_{-1}^*, t)$ be the Glauber–Sudarshan P-functions that describe the fluctuations of the side-mode fields, respectively. In the threshold region, we obtain the following equation for $P(\beta_1, \beta_1^*, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\beta_1,\beta_1^*,t)=\left\{-\frac{\partial}{\partial\beta_1}(\lambda_1\beta_1-a\beta_1|\beta_1|^2)-\frac{\partial}{\partial\beta_1^*}(\lambda_1^*\beta_1^*-a^*\beta_1^*|\beta_1|^2)+D\frac{\partial^2}{\partial\beta_1^*\,\partial\beta_1}\right\}P(\beta_1,\beta_1^*,t), \tag{6}$$

where λ_1 is the eigenvalue of side-mode 1; hence $\mathrm{Re}\,\lambda_1$ is negative (positive) below (above) instability threshold. The real part of a and the diffusion coefficient, D, are positive. The imaginary parts of λ and a shift the side-mode frequency with respect to its empty-cavity value. The function $P(\beta_{-1}, \beta_{-1}^*, t)$ obeys a completely similar equation.

The relevant point is that (6) is formally identical to the well known Risken equation (Risken 1966), which describes the statistics of the laser field in the threshold region. This means that we can immediately exploit all the results obtained for the Risken equation, to describe fluctuations at the onset of self-pulsing behaviour. Most importantly, we can conclude that these fluctuations are as remarkable as laser fluctuations at threshold. More precisely, let us distinguish the static and the dynamical aspects.

First, let us consider the asymptotic, long time self-pulsing régime. If we do not observe the whole electric field, but only a sideband in the instability threshold region we find large fluctuations, as we would expect in a critical situation. For instance, it would be interesting to measure the photon statistics, exactly as was done many years ago for the laser. Furthermore, from our knowledge of the laser case we infer that the sideband spectral line not only exhibits narrowing on approaching the instability threshold, but goes on narrowing beyond threshold.

Next, let us consider the approach to the asymptotic, stationary self-pulsing régime. This approach will present anomalous fluctuations, typical of the decay of the unstable state. These fluctuations can in part counteract the critical slowing down that arises at the instability threshold. This effect is similar to that described by Broggi & Lugiato (this symposium).

5. Concluding remarks

The matter of instabilities and oscillatory behaviour, both periodic and chaotic, in nonlinear systems is more and more attracting the interest of the scientific community. In the special case of optical bistability, this interest also presents practical aspects that arise from the perspective of realizing all-optical clocks, which can be used for example to drive optical transistors.

We think that both types of instabilities discussed in this paper, namely the resonant-mode and the off-resonance mode instabilities, are good candidates for the construction of an optical clock. They arise in mixed absorptive and dispersive o.b. We have demonstrated that they

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persist in the presence of a Gaussian transverse profile, and that they arise for values of the parameters that we consider accessible. The pulsation period is different in the two cases: on the order of the cavity build-up time $\mathscr{L}/cT$ in the resonant mode instability, and on the order of the cavity transit time $\mathscr{L}/c$ in the multimode instability.

We have also shown that the appearance of self-oscillations is accompanied by remarkable critical fluctuations. This on the one hand shows the relevance of fluctuations in the dynamical behaviour of the system, and on the other hand suggests new experimental observations of a statistical type on optical systems.

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